Correlations, spectral gap, and entanglement in harmonic quantum systems on generic lattices

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Abstract

We investigate the relationship between the gap between the energy of the ground state and the first excited state and the decay of correlation functions in harmonic lattice systems. We prove that in gapped systems, the exponential decay of correlations follows for both the ground state and thermal states. Considering the converse direction, we show that an energy gap can follow from algebraic decay and always does for exponential decay. The underlying lattices are described as general graphs of not necessarily integer dimension, including translationally invariant instances of cubic lattices as special cases. Any local quadratic couplings in position and momentum coordinates are allowed for, leading to quasi-free (Gaussian) ground states. We make use of methods of deriving bounds to matrix functions of banded matrices corresponding to local interactions on general graphs. Finally, we give an explicit entanglement-area relationship in terms of the energy gap for arbitrary, not necessarily contiguous regions on lattices characterized by general graphs.

1 Introduction

On physical grounds, one expects that correlations in ground states of gapped many-body systems – quantum systems for which the energy of the lowest exitation is strictly larger than the ground state energy – decay exponentially. This implication of the energy gap to the exponential decay of two-point equal-time correlation functions is a well-established observation on non-critical quantum many-body systems [1, 2, 3, 4, 5, 6, 7, 8, 9]. Yet, surprisingly perhaps, rigorous proofs of this implication for spin-models in higher dimensional cubic or general lattices have been found only very recently. Notably, Ref. [6] reexamines this question for finite-dimensional constituents, which has been generalized to the case of general lattices defined by graphs that may have any non-integer dimension in Refs. [4, 7].

In this paper, we rigorously reconsider this question for a class of systems on general lattices the constituents of which are infinite-dimensional quantum systems: the quasi-free case of harmonic systems on general lattices (see Figure 1). This harmonic case is particularly transparent, as the entire discussion of state properties can be done in terms of the second moments of the states. This case has notably taken a central

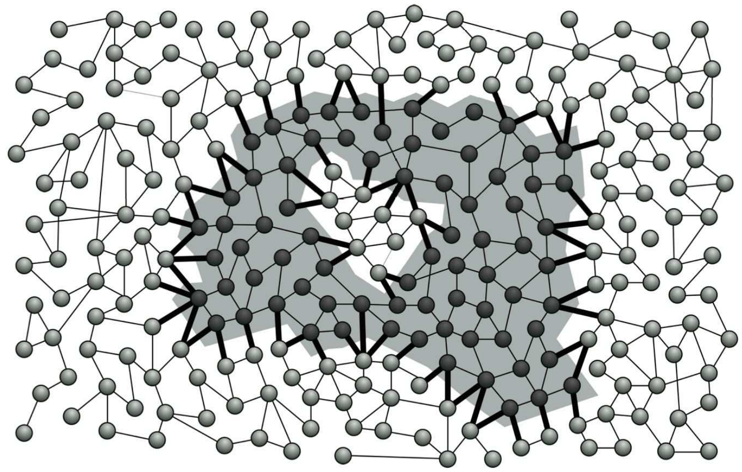

Figure 1: A general graph with a distinguished region $I \subset L$ (gray area, dark gray oscillators). Lines indicate the edge set, the number of bold lines is the surface area $s(I)$.

role in recent discussions of questions of entanglement scaling in many-body systems, as a class of natural physical systems where sophisticated questions on the scaling of entanglement – in form of, e.g., *entanglement-area relationships* [10, 11, 12] and other issues [13, 14, 15, 16, 17] – are relatively accessible.

The general lattices are not assumed to be necessarily translationally invariant, and the underlying graphs may have any spatial dimension. Harmonic systems, needless to say, have been considered many times before, in particular the case of nearest-neighbor interaction [18]: Here, we investigate in a very comprehensive manner the implications of a gap to the two-point correlation functions and the converse direction for all harmonic lattice systems with local Hamiltonians reflecting general couplings in position as well as in momentum, both for the ground state and Gibbs states (non-zero temperature). Notably, these findings enable us to formulate an entanglement-area law for regions of arbitrary shape within the considered context of harmonic systems on general graphs. Such harmonic systems model discrete versions of Klein Gordon fields, vibrational modes of crystal lattices or of ions in a trap, or serve as approximations to not strictly harmonic systems. In contrast to, for example Ref. [4], we will not rely on using variants of Lieb-Robinson [19] bounds on the group velocity to prove the validity of our bounds. Instead we will make use of generalizations of the methods introduced in Ref. [20] to the case of general lattices (see Figure 2).

- We briefly introduce the notion of *quasi-free states as ground or thermal states of harmonic Hamiltonians on generic lattices* described by graphs, with coupling in position and momentum coordinates.

- For *local harmonic Hamiltonians on generic lattices* we prove that whenever the

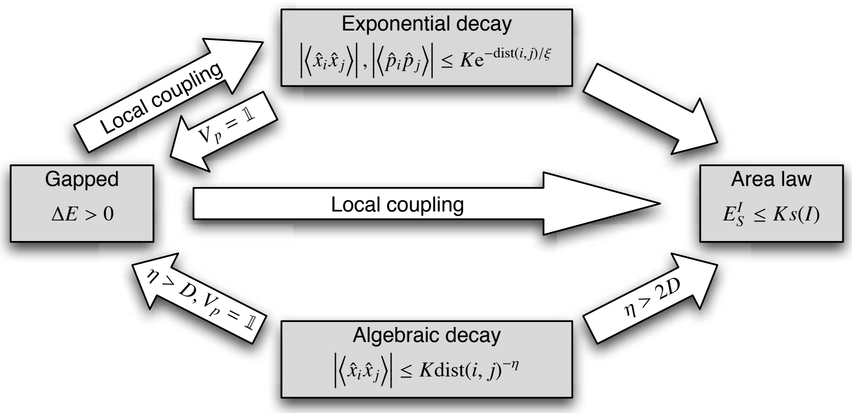

Figure 2: Schematic summary of obtained results.

system is gapped, exponential decay of the correlations in the *ground state* with the canonical coordinates follows (Theorem 1).

- Conversely, for systems (not necessarily locally) coupled in position, we demonstrate that for *sufficiently fast decay of the correlations*, the existence of a spectral gap can be deduced. This includes sufficiently strongly algebraically decaying correlations (Theorem 2,3).
- These findings give rise to a proven *equivalence of exponentially decaying couplings and the existence of a spectral gap* for systems coupled in position on generic lattices (Corollary 1).
- We prove that for gapped harmonic lattice systems on generic lattices in *Gibbs states* (thermal states), exponential decay of the correlations follows (Theorem 4).
- We give a connection [22, 11, 12] to entanglement properties in many-body systems, in this instance of harmonic lattice systems. We show that similarly to Ref. [12], the *entropy of a subsystem of a gapped lattice system is bounded from above by a quantity linear in the boundary area* of the distinguished subsystem, which may be of arbitrary shape (Theorem 5). This generalizes the *area theorem* of Refs. [11, 12] to systems on general lattices.
- We finally discuss several examples in detail, such as instances of *disordered systems* or *thermal states of rotating wave Hamiltonians*. We also investigate a case of *non-local algebraically decaying interactions*.

Note that simultaneously, similar findings were published in Ref. [21], where in contrast to this paper, emphasis was put on ground state properties of translationally invariant harmonic systems on cubic lattices, but where in turn, non-local interactions and critical cases were considered in much more detail.

2 Considered models and main results

2.1 Harmonic systems on general lattices

We consider quadratic bosonic Hamiltonians on generic lattices. This means that the lattice is characterized by general graphs [23]. The set L of vertices with cardinality $|L|$ is associated with the set of sites in the lattice, each of which corresponding to a bosonic degree of freedom. The simple graph $G = (L, E)$ is identified with the lattice, where E is the edge set or, equivalently, the adjacency matrix characterizing a neighborhood relation between physical systems. In a cubic lattice, say, G will represent just this general lattice, regardless of the interaction, which may be finite-ranged beyond nearest or next-to-nearest neighbor interactions.

We will consider paths in the sense of standard graph theory, as sequence of vertices connecting a start and an end vertex, each of which are connected by an edge. For two vertices $i, j \in L$, the integer $\text{dist}(i, j)$ denotes the graph-theoretical distance, so the length of the shortest path connecting i and j [23], with respect to the edge set E.

(i) The Hamiltonian is assumed to be of the form

$$\hat{H} = \hat{p}^T V_p \hat{p} + \hat{x}^T V_x \hat{x}. \tag{1}$$

where $V_x, V_p \in \mathbb{R}^{|L|\times|L|}$ are positive matrices, with $\hat{x}^T = (\hat{x}_1, ..., \hat{x}_{|L|})$ and $\hat{p}^T = (\hat{p}_1, ..., \hat{p}_{|L|})$ being the canonical coordinates satisfying the canonical commutation relations. Hence, we allow for any general coupling in position and momentum coordinates (but no coupling involving both simultaneously). We write $(V_{x,p})_{i,j}$, $i, j = 1, ..., |L|$, to label the element of the matrices $V_{x,p}$ that belongs to the coupling between two vertices i, j, respectively. For simplicity of notation, these couplings in position and momentum can be subsumed into the tuple $C = (G, V_x, V_p)$. Whenever two sites $i, j \in L$ are coupled in position, the element $(V_x)_{i,j}$ will be non-zero, and similarly for the momentum coordinates. The range of the interaction is taken with respect to the metric dist$(\cdot, \cdot)$, such that V_x and V_p inherit a neighborhood relation.

Note that V_x and V_p can in turn be conceived as adjacency matrices of a weighted graph with the same vertex set L, but a different edge set. The above description is similar to the assessment of the correlation function for generic spin systems in Refs. [4, 7]. Note that such bosonic harmonic lattice systems resemble to some extend the concept of graph states for spin or qubit systems in the sense of Refs. [24, 25], compare also Refs. [12, 14, 16].

The dimension of the underlying lattice G can take any positive value in the following sense: We may define a sphere $S_r(i)$ for some $i \in L$, centered at site i with integer radius r as

$$S_r(i) := \{l \in L : \text{dist}(l, i) = r\}\,. \tag{2}$$

Then there exists a smallest $d > 0$ of the lattice, notably not necessarily integer [7], such that for all $r \in \mathbb{N}$,

$$\sup_{i\in L} |S_r(i)| \leq c r^{d-1} \tag{3}$$

for some $c > 0$. This number d is taken as the dimension of the lattice. Note that for cubic lattices this dimension coincides with the natural underlying spatial dimension, and the above expression is the same for open and periodic boundary conditions. Similarly we can define a ball centered at site $i \in L$ with radius r as

$$B_r(i) := \{l \in L : \text{dist}(l, i) \leq r\} \tag{4}$$

and can express the maximal volume

$$v_{d,r} := \sup_{i \in L} |B_r(i)| \leq c \sum_{j=1}^{r} j^{d-1} \tag{5}$$

of a ball with radius r in the graph theoretical sense. For a consideration of the dimension of general graphs and the discussion of self-similarity in this context, see, e.g., Ref. [26]. For our purposes, the dimension will only enter the bounds through the volume of a ball of some radius.

Concerning the locality, we assume the following:

(ii) Most Hamiltonians that we consider are local, corresponding to a finite-ranged interaction, if not otherwise specified. This means that there exists an $m \in \mathbb{N}$, such that for all $i \in L$

$$(V_x)_{i,j} = 0, \ (V_p)_{i,j} = 0 \tag{6}$$

for all $j \in L$ for which

$$\text{dist}(i, j) > m/2. \tag{7}$$

Note that this in turn means that the Hamiltonian is of the form as in Eq. (1) and can be written as

$$\hat{H} = \sum_{\substack{X \subset L, \\ \text{diam}(X) \leq m/2}} \hat{h}_X, \tag{8}$$

$X \subset L$ being subsets the diameter of which satisfy

$$\text{diam}(X) = \sup_{i,j \in X} \text{dist}(i, j) \leq m/2. \tag{9}$$

It will turn out to be convenient to collect the $|L|$ conjugate pairs of canonical coordinates in a vector $\hat{r}^T = (\hat{x}_1, ..., \hat{x}_{|L|}, \hat{p}_1, ..., \hat{p}_{|L|})$, the entries of which satisfying the canonical commutation relations (CCR), giving rise to a symplectic scalar product [27]. The Hamiltonian

$$\hat{H} = \hat{r}^T \begin{pmatrix} V_x & 0 \\ 0 & V_p \end{pmatrix} \hat{r}, \tag{10}$$

can now be brought into diagonal form by means of a symplectic transformation, so by means of linear transformations $S \in Sp(2|L|, \mathbb{R})$ preserving the symplectic form [27], defined by the skew-symmetric symplectic matrix

$$\Sigma = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}. \tag{11}$$

Let $O \in O(|L|)$ be the orthogonal matrix that brings $V_x^{1/2} V_p V_x^{1/2}$ into diagonal form, $O^T V_x^{1/2} V_p V_x^{1/2} O =: D = \text{diag}(d_1, \cdots, d_{|L|})$. Then

$$S = \begin{pmatrix} V_x^{-1/2} O & 0 \\ 0 & V_x^{1/2} O \end{pmatrix} \in Sp(2|L|, \mathbb{R}) \tag{12}$$

and with $\hat{r} = S\hat{r}'$, the Hamiltonian finally takes the form

$$\hat{H} = 2 \sum_{i=1}^{|L|} d_i^{1/2} \left(\hat{a}_i^\dagger \hat{a}_i + 1/2 \right), \quad \hat{a}_i := \frac{\hat{x}_i' + \text{i} d_i^{1/2} \hat{p}_i'}{(2 d_i^{1/2})^{1/2}}. \tag{13}$$

The ground state energy is thus given by

$$E_0 = \sum_{i=1}^{|L|} d_i^{1/2} = \text{tr}[(V_x^{1/2} V_p V_x^{1/2})^{1/2}] = \text{tr}[(V_x V_p)^{1/2}]. \tag{14}$$

The energy of the first excited state is given by

$$E_1 = E_0 + \Delta E, \tag{15}$$

$$\Delta E := 2 \lambda_{\text{min}}^{1/2} (V_x^{1/2} V_p V_x^{1/2}) = 2 \lambda_{\text{min}}^{1/2} (V_x V_p), \tag{16}$$

where we denote by $\lambda_{\text{min}}(A)$ the minimum eigenvalue of a matrix A. The corresponding ground state of such a Hamiltonian is a quasi-free (Gaussian) state [27], completely characterized by the first and second moments. Quasi-free bosonic or Gaussian state means that the characteristic function – expectation value of the Weyl displacement operator – is a Gaussian function in state space. Equivalently, a state is Gaussian if its Wigner function is a Gaussian function in state space. The second moments can be embodied in the covariance matrix with respect to the ground state

$$\begin{aligned} \gamma'_{i,j} &= \left\langle \{ (\hat{r}_i' - \langle \hat{r}_i' \rangle), (\hat{r}_j' - \langle \hat{r}_j' \rangle) \}_+ \right\rangle \\ &= \langle \hat{r}_i' \hat{r}_j' \rangle + \langle \hat{r}_j' \hat{r}_i' \rangle. \end{aligned} \tag{17}$$

Here, first moments vanish as the ground state is the vacuum. We find

$$\gamma' = D^{1/2} \oplus D^{-1/2}, \tag{18}$$

and in the original coordinates we have

$$\begin{aligned} \gamma &= S \gamma' S^T \\ &= \left(V_x^{-1/2} (V_x^{1/2} V_p V_x^{1/2})^{1/2} V_x^{-1/2} \right) \oplus \left(V_x^{1/2} (V_x^{1/2} V_p V_x^{1/2})^{-1/2} V_x^{1/2} \right). \end{aligned} \tag{19}$$

This expression simplifies significantly in case of commuting V_x and V_p. Often, $V_p = \mathbb{1}$, in which case

$$\gamma = V_x^{-1/2} \oplus V_x^{1/2}, \tag{20}$$

see Ref. [10]. The language chosen here is the one used also in the assessment of entanglement properties of harmonic chains [10, 13] or more general harmonic systems on lattices [11, 12, 27, 14, 15, 28], mildly generalized to coupling in both position and momentum and to general lattices.

2.2 Exponential decay of correlations in ground states

The first result links the spectral gap of the Hamiltonian to the exponentially decaying correlation functions of the system. Note that the bound on the right hand side depends only on the dimension of the underlying graph, the gap, the parameter characterizing the range of the interactions, and bounds to the coupling strength.

Theorem 1 (Exponentially decaying correlation functions) *Consider a Hamiltonian on a general lattice of dimension d with a coupling $C = (G, V_x, V_p)$ of finite range m as defined above. Then for all $i, j \in L$ with dist$(i, j) \geq m$ the ground state satisfies*

$$|\langle \hat{x}_i \hat{x}_j \rangle| \leq K \|V_p\| \exp\left[-\mathit{dist}(i,j)/\xi\right], \tag{21}$$

$$|\langle \hat{p}_i \hat{p}_j \rangle| \leq K \|V_x\| \exp\left[-\mathit{dist}(i,j)/\xi\right], \tag{22}$$

where

$$K := \frac{\|V_x V_p\|^{1/2} v_{d,m/2}}{(\Delta E/2)^2}, \quad \xi := \frac{2m}{\log\left(\frac{\|V_x V_p\|}{\|V_x V_p\| - (\Delta E/2)^2}\right)}, \tag{23}$$

where $\|\cdot\|$ denotes the operator norm and $\Delta E = 2\lambda_{\mathit{min}}^{1/2}(V_x V_p)$ is the energy difference between the first excited state and the ground state.

Note that if the energy is expressed in units of $\|V_x V_p\|$, then the correlation length depends only on the gap and the range of the interactions.

2.3 Spectral gap from algebraically decaying correlation functions

In this subsection, we consider an instance of the converse direction, compare Ref. [2]: we assume that the system has algebraically decaying correlation functions in the position coordinates and assume that we have a – not necessarily local – coupling in these coordinates and $V_p = \mathbb{1}$, then we can conclude that the system must be gapped if the decay is sufficiently strong. This is made more rigorous in the subsequent theorem.

Theorem 2 (Spectral gap from algebraic decay) *Consider a sequence of couplings $C^{(n)} = (G^{(n)}, V^{(n)}, \mathbb{1})$, $n \in \mathbb{N}$, on general lattices $G^{(n)} = (L^{(n)}, E^{(n)})$ of dimension $d^{(n)}$. Let $K \geq 0$, $K_0 \geq 0$, $\eta > \sup d^{(n)} =: d$, and $c := \sup c^{(n)} < \infty$ as defined in Eq. (3). If the ground states satisfy*

$$|\langle \hat{x}_i^{(n)} \hat{x}_j^{(n)} \rangle| \leq \begin{cases} K_0 & \text{for } i = j, \\ K\,\mathit{dist}(i,j)^{-\eta} & \text{for } i \neq j, \end{cases} \tag{24}$$

for all $i, j \in L^{(n)}$ and for all n, then the energy difference between the first excited and the ground state satisfies

$$\inf \Delta E^{(n)} =: \Delta E \geq \frac{2}{K_0 + cK\zeta(1 + \eta - d)} > 0, \tag{25}$$

where ζ is the Riemann zeta function.

Note that, if in addition to the assumptions of the above theorem the coupling $V^{(n)}$ is (i) of finite range m for all n, and (iia) the momentum correlations decay as in Eqs. (24) or alternatively, (iib) the coupling has a finite coupling strength, $\sup \|V^{(n)}\| < \infty$,[1] it immediately follows from Theorem 1 that in fact the correlations decay exponentially. This is an interesting observation, showing that in a sense the class of harmonic lattice systems is not rich enough to – roughly speaking – show all "types of decays" of the correlations.

2.4 Equivalence of spectral gap and exponentially decaying correlation functions

For exponentially decaying correlations we can always conclude that the system is gapped. This statement can be formulated as follows.

Theorem 3 (Spectral gap from exponential decay) *Let $K \geq 0$ and $\xi \geq 0$, and consider a sequence of couplings $C^{(n)} = (G^{(n)}, V^{(n)}, \mathbb{1})$, $n \in \mathbb{N}$, on general lattices $G^{(n)} = (L^{(n)}, E^{(n)})$ of dimension $d^{(n)}$, $d := \sup d^{(n)} < \infty$, and $c := \sup c^{(n)} < \infty$. If the ground state satisfies*

$$|\langle \hat{x}_i^{(n)} \hat{x}_j^{(n)} \rangle| \leq K \exp\left[-\mathit{dist}(i,j)/\xi\right] \tag{26}$$

for all $i, j \in L^{(n)}$ and all n, then the energy difference between the first excited and the ground state satisfies

$$\inf \Delta E^{(n)} =: \Delta E \geq \frac{2}{K(1 + c Li_{1-d}(\mathrm{e}^{-1/\xi}))} > 0, \tag{27}$$

where Li_{1-d} is the polylogarithm of degree $1-d$.

Together with Theorem 1 this establishes for locally coupled systems the following equivalence.

Corollary 1 (Equivalence of spectral gap and exponentially decaying correlations) *Consider the sequence of couplings $C^{(n)} = (G^{(n)}, V^{(n)}, \mathbb{1})$, $n \in \mathbb{N}$, of finite range m on general lattices $G^{(n)} = (L^{(n)}, E^{(n)})$ of dimension $d^{(n)}$ with $d := \sup d^{(n)} < \infty$, and $c := \sup c^{(n)} < \infty$. Then the following statements are equivalent.*

(i) There exist constants $K, \xi > 0$ such that both correlations with respect to the ground state satisfy

$$|\langle \hat{x}_i^{(n)} \hat{x}_j^{(n)} \rangle|, |\langle \hat{p}_i^{(n)} \hat{p}_j^{(n)} \rangle| \leq K \exp\left[-\mathit{dist}(i,j)/\xi\right] \tag{28}$$

for all $i, j \in L^{(n)}$ and all n.

(ii) The energy difference between the first excited state and the ground state satisfies

$$\inf \Delta E^{(n)} =: \Delta E > 0 \tag{29}$$

and the coupling is of finite strength, $\sup \|V^{(n)}\| < \infty$.

[1]Note that (iib) follows from (iia) using the same methods as in the proof for Theorem 2.

This result establishes that in local harmonic systems on generic lattices, being non-critical in the sense of being gapped is equivalent with finding exponentially decaying correlation functions.

2.5 Expontial decay in Gibbs states

In this subsection, we consider thermal Gibbs states [27] corresponding to some temperature $T > 0$,

$$\varrho(T) = \frac{\exp(-\hat{H}/T)}{\mathrm{tr}[\exp(-\hat{H}/T)]}. \tag{30}$$

Such states are again quasi-free (Gaussian) states and can thus also be uniquely characterized by their covariance matrices, appropriately modified for this case of non-zero temperature. For a study of two-point correlations in Fermi-systems at non-zero temperature see also Ref. [5]. In the diagonal basis (13), it takes the form

$$\begin{aligned}\gamma(T)' &= D^{1/2}\left[\mathbb{1} + 2\big(\exp(2D^{1/2}/T) - \mathbb{1}\big)^{-1}\right] \\ &\quad \oplus D^{-1/2}\left[\mathbb{1} + 2\big(\exp(2D^{1/2}/T) - \mathbb{1}\big)^{-1}\right]\end{aligned} \tag{31}$$

and in original coordinates we have

$$\begin{aligned}\gamma(T) = S\gamma(T)'S^T &= \gamma(0) + V_x^{-1/2} G \big(V_x^{1/2} V_p V_x^{1/2}\big)^{1/2} V_x^{-1/2} \\ &\quad \oplus V_x^{1/2}\big(V_x^{1/2} V_p V_x^{1/2}\big)^{-1/2} G V_x^{1/2}, \qquad (32)\\ G &:= 2\left(\exp\left(2(V_x^{1/2} V_p V_x^{1/2})^{1/2}/T\right) - \mathbb{1}\right)^{-1}. \qquad (33)\end{aligned}$$

In the following we will now assume that matrices V_x and V_p commute. This yields a simplified covariance matrix

$$\begin{aligned}\gamma(T) &= \gamma(0) + \big(V_x^{-1/2} V_p^{1/2} G\big) \oplus \big(V_x^{1/2} V_p^{-1/2} G\big), \qquad (34)\\ G &= 2\left(\exp\big(2(V_x V_p)^{1/2}/T\big) - \mathbb{1}\right)^{-1}. \qquad (35)\end{aligned}$$

Furthermore, we require the following assumption on the lattice. This assumption is very similar to the ones in Ref. [4], and is satisfied for a large class of natural lattice systems.

Assumption 1 (Lattice structure) *Consider a Hamiltonian on a general lattice $G = (L, E)$ with couplings $C = (G, V_x, V_p)$ as in Theorem 1. Then, it is assumed that there exist constants $l_0 \geq 0$ and $\nu > 0$ such that*

$$\sum_{k\in L} \exp(-\mu\, \mathit{dist}(i,k)) \exp(-\mu\, \mathit{dist}(k,j)) \leq l_0 \exp(-\nu\, \mathit{dist}(i,j)), \tag{36}$$

for all $i, j \in L$ and for

$$\mu := \frac{\log\left(\frac{\|V_x V_p\|}{\|V_x V_p\| - (\Delta E/2)^2}\right)}{m}. \tag{37}$$

For example, for a cubic lattice in d dimensions, we have that $L = [1, ..., n]^{\times d}$, and $\text{dist}(i, j) = \sum_{\delta=1}^{d} |i_\delta - j_\delta|$. Therefore, we arrive at

$$\sum_{k \in L} \mathrm{e}^{-\mu \text{dist}(i,k)} \mathrm{e}^{-\mu \text{dist}(k,j)} = \prod_{\delta=1}^{d} \sum_{k_\delta=1}^{n} \mathrm{e}^{-\mu|i_\delta - k_\delta|} \mathrm{e}^{-\mu|k_\delta - j_\delta|}$$

$$= \prod_{\delta=1}^{d} \left(\mathrm{e}^{-\mu|i_\delta - k_\delta|} \left(|i_\delta - j_\delta| + \frac{\mathrm{e}^{2\mu}+1}{\mathrm{e}^{2\mu}-1} \right) - \frac{\mathrm{e}^{-\mu(i_\delta + j_\delta - 2)} + \mathrm{e}^{-\mu(2n - i_\delta - j_\delta)}}{\mathrm{e}^{2\mu}-1} \right)$$

$$\leq \left(\frac{2}{\mu \mathrm{e}} + \frac{\mathrm{e}^{2\mu}+1}{\mathrm{e}^{2\mu}-1} \right)^{d} \exp(-\mu \text{dist}(i,j)/2), \quad (38)$$

i.e., in this cubic case, we have $\nu = \mu/2$.

Under the previous assumption, one can arrive at the subsequent statement on two-point correlations in Gibbs states. Note that it is not merely trivially true that the correlations are shorter-ranged in thermal states: there are examples of Hamiltonians where Gibbs states at higher temperatures have longer-ranged correlations than at zero temperature, see Example 2.

Theorem 4 (Correlations in systems at finite temperature) *Consider a finite-ranged Hamiltonian corresponding to $C = (G, V_x, V_p)$, $[V_x, V_p] = 0$, on a general lattice $G = (L, E)$ with assumptions as in Theorem 1 and equipped with Assumption 1. Then, for dist$(i, j) \geq m$, the Gibbs state with respect to some temperature T satisfies*

$$|\langle \hat{x}_i \hat{x}_j \rangle| \leq K(T) \|V_p\| \exp\left[-\textit{dist}(i,j)/\xi\right], \quad (39)$$

$$|\langle \hat{p}_i \hat{p}_j \rangle| \leq K(T) \|V_x\| \exp\left[-\textit{dist}(i,j)/\xi\right], \quad (40)$$

where

$$K(T) := \frac{\|V_x V_p\|^{1/2} v_{d,m/2}}{(\Delta E/2)^2} \left(1 + \frac{4 l_0 \|V_x V_p\| / (\Delta E/2)^2}{\exp\left(\frac{\Delta E}{T} \left(1 - \frac{(\Delta E/2)^2}{4\|V_x V_p\|} \right)^{1/2} \right) - 1} \right), \quad (41)$$

$$\xi := \max \left\{ \frac{2}{\nu}, \frac{2m}{\log\left(\frac{\|V_x V_p\|}{\|V_x V_p\| - (\Delta E/2)^2} \right)} \right\}. \quad (42)$$

This theorem ends the list of main statements on the relation between the system being gapped and the decay of the two-point correlation functions with respect to the ground state and thermal states. In the next section we will study the implications of the above findings on the entanglement scaling of distinguished regions of general lattices

2.6 Entanglement scaling in ground states of harmonic systems on general lattices

The above statements on the decay of correlation functions have immediate implications on the scaling of entanglement. More specifically, in a lattice system we may

distinguish a certain part $I \subset L$ and ask for the degree of entanglement between the degrees of freedom of this region and the rest of the lattice. This question is that of the scaling of the *geometric entropy of this region*. This question goes back to seminal numerical investigations in Refs. [31, 32] (see also Ref. [33]), suggesting a linear relationship between the geometric entropy and the boundary area of a distinguished region. In turn, in Refs. [11, 12] a rigorous relationship between the boundary area and the entropy of the region in cubic harmonic lattice systems has first been analytically established.

In fact, in the light of the above findings, one may infer the validity of such an 'area theorem' for general lattices, not only for cubic lattices, exploiting exactly the same methods of proof. Defining the surface area of a distinguished region I of the whole lattice as

$$s(I) := \sum_{i \in L \setminus I} \sum_{\substack{j \in I \\ \mathrm{dist}(i,j)=1}} 1, \tag{43}$$

see Figure 1, one arrives at a bound to the von-Neumann entropy,

$$E_S^I = S(\varrho_I) = -\mathrm{tr}\left[\varrho_I \log_2(\varrho_I)\right], \tag{44}$$

of the reduced ground state, $\varrho_I = \mathrm{tr}_{L \setminus I}[\varrho]$, with respect to the distinguished region.

Theorem 5 (Entanglement-area law for general lattices) *Consider a general lattice $G = (L, E)$ of dimension d equipped with a coupling $(G, V, \mathbb{1})$. If the coupling is of finite range m, the entropy of entanglement satisfies*

$$E_S^I \leq \frac{4\|V\| c^2 Li_{1-2d}(\mathrm{e}^{-1/\xi})}{\log(2)(\Delta E/2)^2} s(I), \tag{45}$$

where Li_{1-2d} is the polylogarithm of degree $1-2d$, ΔE the energy gap above the ground state, $s(I)$ the surface area of I, c as defined in Eq. (3), and

$$\xi := \frac{m}{\log\left(\frac{\|V\|}{\|V\| - (\Delta E/2)^2}\right)}. \tag{46}$$

In this form, the contribution of a geometrical factor, as well as one originating from the correlation length, is very transparent. Note that, using Theorem 4 and the bounds derived in Ref. [12], one arrives at a similar bound for the distillable entanglement of thermal states.

Indeed, this theorem covers the entanglement-area relationship in all generality for gapped harmonic models on generic lattices. In particular, for cubic regions with volume L^d in d dimensions this means that the geometric entropy is bounded by expressions linear in L^{d-1}. For critical fermionic quasi-free systems, in turn, there are instances where one finds a behavior of $L^{d-1} \log L$ for the geometric entropy [34, 35]. In contrast, there is numerical evidence that for bosonic harmonic systems, even in the critical case, the validity of the entanglement-area relationship of L^{d-1} is preserved [36, 37]. To strictly prove or refute this relationship in this critical case constitutes one of the intriguing open problems in the field.

3 Proofs

This section will contain the proofs of the statements made before. We will make extended use of the bandedness of the interaction matrix in the sense of the metric dist$(\cdot,\cdot)$. The main ingredient will be polynomial approximations to matrix functions of matrices. We generalize a statement of Ref. [20] to the case of general lattices. This extends the generalization put forth in Ref. [12] for general cubic lattices. Note that for statements of the type following, random walk methods would similarly be suitable. In particular, for the case of nearest-neighbor interaction, this analysis has been done, see, e.g., Ref. [18] for a comprehensive introduction.

In the following we will be needing the subsequent lemma which is concerned with the range of matrices in the sense of dist$(\cdot,\cdot)$. Note that the matrix power is taken in the sense of the ordinary matrix power.

Lemma 1 (Range of matrix powers) *Let the $|L|\times|L|$ matrix A be a coupling matrix on a general lattice $G=(L,E)$. If A is of finite range m, i.e., $A_{i,j}=0$ for dist$(i,j)>m/2$, $m\in\mathbb{N}$, then A^n, $n\in\mathbb{N}$, is of range nm.*

Proof. This can be proven by induction over n. For $n=1$ we have $A^n=A$ is of range nm. Now suppose A^n has range nm, i.e., we have $(A^n)_{i,j}=0$ for dist$(i,j)>nm/2$. Now,

$$\left(A^{n+1}\right)_{i,j}=\sum_{k\in L}\left(A^n\right)_{i,k}A_{k,j}. \tag{47}$$

Suppose dist$(i,j)>m(n+1)/2$. It follows that for dist$(i,k)>nm/2$ we have $(A^n)_{i,k}=0$ as A^n is of range nm. For dist$(i,k)\leq nm/2$ we have

$$m(n+1)/2<\text{dist}(i,j)\leq\text{dist}(i,k)+\text{dist}(k,j)\leq nm/2+\text{dist}(k,j), \tag{48}$$

i.e., dist$(k,j)>m/2$, i.e., $A_{k,j}=0$. In turn, for dist$(k,j)\leq m/2$ we have

$$m(n+1)/2<\text{dist}(i,j)\leq\text{dist}(i,k)+\text{dist}(k,j)\leq\text{dist}(i,k)+m/2, \tag{49}$$

i.e., dist$(i,k)>nm/2$, i.e., $(A^n)_{i,k}=0$ as A^n is of range nm. Finally, we hence arrive at $(A^{(n+1)})_{i,j}=0$ for dist$(i,j)>m(n+1)/2$, which concludes the proof. □

3.1 Proof of Theorem 1

From the covariance matrix γ, Eq. (19), we have

$$\langle\hat p_i\hat p_j\rangle = \left(V_x^{1/2}(V_x^{1/2}V_pV_x^{1/2})^{-1/2}V_x^{1/2}\right)_{i,j}, \tag{50}$$

$$\begin{aligned}\langle\hat x_i\hat x_j\rangle &= \left(V_x^{-1/2}(V_x^{1/2}V_pV_x^{1/2})^{1/2}V_x^{-1/2}\right)_{i,j}\\ &= \left(V_p^{1/2}(V_p^{1/2}V_xV_p^{1/2})^{-1/2}V_p^{1/2}\right)_{i,j}.\end{aligned} \tag{51}$$

We can now compute the matrix $(V_x^{1/2}V_pV_x^{1/2})^{-1/2}$ and similarly $(V_p^{1/2}V_xV_p^{1/2})^{-1/2}$ using the power-series expansion of the square root, which is valid for $|x| < 1$,

$$(1-x)^{-1/2} = 1 + \sum_{k=1}^{\infty} a_k x^k, \tag{52}$$

where the coefficients a_k are bounded by $a_k \leq 1$.

$$\begin{aligned}\left(V_x^{1/2}V_pV_x^{1/2}\right)^{-1/2} &= \|V_xV_p\|^{-1/2}\left(\mathbb{1} - (\mathbb{1} - V_x^{1/2}V_pV_x^{1/2}/\|V_xV_p\|)\right)^{-1/2} \\ &= \|V_xV_p\|^{-1/2}\left(\mathbb{1} + \sum_{k=1}^{\infty} a_k O\left(\mathbb{1} - D/\|V_xV_p\|\right)^k O^T\right). \end{aligned}\tag{53}$$

Now,

$$D = O^T V_x^{1/2}V_pV_x^{1/2}O = \left(V_x^{-1/2}O\right)^{-1} V_pV_x \left(V_x^{-1/2}O\right), \tag{54}$$

i.e.,

$$\left(V_x^{1/2}V_pV_x^{1/2}\right)^{-1/2} = \|V_xV_p\|^{-1/2}V_x^{1/2}W_{p,x}V_x^{-1/2}, \tag{55}$$

$$W_{p,x} := \left(\mathbb{1} + \sum_{k=1}^{\infty} a_k\left(\mathbb{1} - (V_pV_x)/\|V_xV_p\|\right)^k\right) \tag{56}$$

and similarly

$$\left(V_p^{1/2}V_xV_p^{1/2}\right)^{-1/2} = \|V_xV_p\|^{-1/2}V_p^{1/2}W_{x,p}V_p^{-1/2}. \tag{57}$$

Note that clearly, $(V_x^{1/2}V_pV_x^{1/2})^{-1/2}$ is a positive matrix, whereas $W_{p,x}$ is in general not a symmetric matrix (and the matrix power is the k-fold concatenation of matrix multiplication). Thus, the correlation functions take the form

$$\langle \hat{x}_i\hat{x}_j\rangle = \|V_xV_p\|^{-1/2}\left(V_pW_{p,x}\right)_{i,j}, \tag{58}$$

$$\langle \hat{p}_i\hat{p}_j\rangle = \|V_xV_p\|^{-1/2}\left(V_xW_{x,p}\right)_{i,j}. \tag{59}$$

Assuming V_x and V_p to be of range $m/2$, i.e., $(V_x)_{i,j} = (V_p)_{i,j} = 0$ for $\text{dist}(i,j) > m/2$, we have that V_xV_p and V_pV_x are of range m. Note that for a matrix to be of finite range in the sense of the above Lemma it is not required that it is symmetric. Hence we can conclude that $(V_xV_p)^k$ and $(V_pV_x)^k$ are of range km. Thus

$$\left|\left(W_{p,x}\right)_{i,j}\right| = \left|\delta_{i,j} + \sum_{k\geq\lceil\text{dist}(i,j)/m\rceil} a_k\left(\left(\mathbb{1} - (V_pV_x)/\|V_xV_p\|\right)^k\right)_{i,j}\right|. \tag{60}$$

For the diagonal terms we have $|(W_{p,x})_{i,i}| \leq \|W_{p,x}\| = \|V_x V_p\|^{1/2}/(\Delta E/2)$ and for the off-diagonal terms we find

$$\begin{aligned}\left|(W_{p,x})_{i,j}\right| &\leq \sum_{k \geq \lceil \text{dist}(i,j)/m \rceil} a_k \left\|\mathbb{1} - (V_p V_x)/\|V_x V_p\|\right\|^k \\ &\leq \sum_{k \geq \lceil \text{dist}(i,j)/m \rceil}^{\infty} \left(1 - \frac{(\Delta E/2)^2}{\|V_x V_p\|}\right)^k . \qquad (61)\end{aligned}$$

The same bound holds for the entries of $W_{x,p}$, i.e., for all $i, j \in L$ we have

$$\left|(W_{p,x})_{i,j}\right|, \left|(W_{x,p})_{i,j}\right| \leq \frac{\|V_x V_p\|}{(\Delta E/2)^2} \left(1 - \frac{(\Delta E/2)^2}{\|V_x V_p\|}\right)^{\text{dist}(i,j)/m} . \qquad (62)$$

For the correlation function $\langle \hat{x}_i \hat{x}_j \rangle$ this yields

$$\begin{aligned}|\langle \hat{x}_i \hat{x}_j \rangle| &\leq \frac{\|V_x V_p\|^{1/2}}{(\Delta E/2)^2} \sum_{k \in L} |(V_p)_{i,k}| \left(1 - \frac{(\Delta E/2)^2}{\|V_x V_p\|}\right)^{\text{dist}(k,j)/m} \\ &= \frac{\|V_x V_p\|^{1/2}}{(\Delta E/2)^2} \sum_{\substack{k \in L, \\ \text{dist}(k,i) \leq m/2}} |(V_p)_{i,k}| \left(1 - \frac{(\Delta E/2)^2}{\|V_x V_p\|}\right)^{\text{dist}(k,j)/m} . \qquad (63)\end{aligned}$$

Now $\text{dist}(i,j) \leq \text{dist}(i,k) + \text{dist}(k,j) \leq m/2 + \text{dist}(k,j)$, and thus we find for $\text{dist}(i,j) \geq m$

$$\begin{aligned}|\langle \hat{x}_i \hat{x}_j \rangle| &\leq \frac{\|V_x V_p\|^{1/2}}{(\Delta E/2)^2} \sum_{\substack{k \in L, \\ \text{dist}(k,i) \leq m/2}} |(V_p)_{i,k}| \left(1 - \frac{(\Delta E/2)^2}{\|V_x V_p\|}\right)^{\text{dist}(i,j)/(2m)} \\ &\leq \frac{\|V_x V_p\|^{1/2}}{(\Delta E/2)^2} \left(1 - \frac{(\Delta E/2)^2}{\|V_x V_p\|}\right)^{\text{dist}(i,j)/(2m)} \|V_p\| \sum_{\substack{k \in L, \\ \text{dist}(k,i) \leq m/2}} 1 . \qquad (64)\end{aligned}$$

For $|\langle \hat{p}_i \hat{p}_j \rangle|$ one can argue in the same manner. Noting that the sum on the right hand side is just $|B_{m/2}(i)|$, we finally have for $\text{dist}(i,j) \geq m$

$$|\langle \hat{x}_i \hat{x}_j \rangle| \leq \frac{\|V_x V_p\|^{1/2} v_{d,m/2}}{(\Delta E/2)^2} \|V_p\| \left(1 - \frac{(\Delta E/2)^2}{\|V_x V_p\|}\right)^{\text{dist}(i,j)/(2m)} , \qquad (65)$$

$$|\langle \hat{p}_i \hat{p}_j \rangle| \leq \frac{\|V_x V_p\|^{1/2} v_{d,m/2}}{(\Delta E/2)^2} \|V_x\| \left(1 - \frac{(\Delta E/2)^2}{\|V_x V_p\|}\right)^{\text{dist}(i,j)/(2m)} , \qquad (66)$$

which concludes the proof of Theorem 1. □

3.2 Proof of Theorem 2

For each n, the energy gap is given by

$$\Delta E^{(n)} = 2\lambda_{\min}((V^{(n)})^{1/2}) = \frac{2}{\|(V^{(n)})^{-1/2}\|} . \qquad (67)$$

The operator norm is bounded from above my the norm corresponding to the maximum sum of elements of any row of the matrix, and hence we may write

$$\begin{aligned} \|(V^{(n)})^{-1/2}\| &\leq \sup_{i\in L^{(n)}} \sum_{j\in L^{(n)}} |((V^{(n)})^{-1/2})_{i,j}| \\ &\leq \sup_{i\in L^{(n)}} \left(K_0 + K \sum_{\substack{j\in L^{(n)} \\ j\neq i}} \mathrm{dist}(i,j)^{-\eta} \right) \end{aligned} \tag{68}$$

and hence

$$\|(V^{(n)})^{-1/2}\| \leq \sup_{i\in L^{(n)}} \left(K_0 + K \sum_{r=1}^{\infty} r^{-\eta} \sum_{\mathrm{dist}(i,j)=r} 1 \right), \tag{69}$$

and therefore

$$\begin{aligned} \|(V^{(n)})^{-1/2}\| &\leq \sup_{i\in L^{(n)}} \left(K_0 + K \sum_{r=1}^{\infty} r^{-\eta} |S_r(i)| \right) \\ &\leq \left(K_0 + K c^{(n)} \sum_{r=1}^{\infty} r^{d^{(n)}-1-\eta} \right) \\ &= K_0 + c^{(n)} K \zeta(1+\eta-d^{(n)}), \end{aligned} \tag{70}$$

where ζ is the Riemann zeta function, and $c^{(n)} > 0$ is defined as in Eq. (3). Thus, we can conclude that

$$\Delta E^{(n)} \geq \frac{2}{K_0 + c^{(n)} K \zeta(1+\eta-d^{(n)})} \geq \frac{2}{K_0 + cK\zeta(1+\eta-d)} > 0, \tag{71}$$

independent of n. □

3.3 Proof of Theorem 3

For each n, the energy gap is given by Eq. (67). Analogous to the previous proof, we have that

$$\begin{aligned} \|(V^{(n)})^{-1/2}\| &\leq K \sup_{i\in L^{(n)}} \left(1 + \sum_{r=1}^{\infty} \exp(-r/\xi) |S_r(i)| \right) \\ &\leq K \left(1 + c^{(n)} \sum_{r=1}^{\infty} \exp(-r/\xi) r^{d^{(n)}-1} \right) \\ &\leq K(1 + cLi_{1-d}(\mathrm{e}^{-1/\xi})) < \infty, \end{aligned} \tag{72}$$

where Li_{1-d} is the polylogarithm of degree $1-d$. As the above bound holds for all n, we can conclude that

$$\inf \Delta E^{(n)} \geq \frac{2}{K(1 + cLi_{1-d}(\mathrm{e}^{-1/\xi}))} > 0. \tag{73}$$

□

3.4 Proof of Theorem 4

In the finite-temperature case, the covariance matrix $\gamma(T)$ is given by

$$\gamma(T) = \gamma(0) + \Big(V_p(V_xV_p)^{-1/2}G\Big) \oplus \Big(V_x(V_xV_p)^{-1/2}G\Big) . \tag{74}$$

We will now proceed by demonstrating the exponential decay of the entries of the matrices G and $(V_xV_p)^{-1/2} = (V_x^{1/2}V_pV_x^{1/2})^{-1/2}$ (V_x and V_p are assumed to be commuting) and will then apply the assumption on the lattice to finally arrive at Theorem 2. To show the exponential decay, we will now proof a generalization of a theorem of Ref. [20] to matrices reflecting general lattices. This latter work is concerned with the exponential decay of entries of matrix functions of banded matrices. The proof will be very similar to the one in Ref. [20], only that the notion of a distance is different: the distance from the main diagonal of a matrix versus dist$(\cdot,\cdot)$ in the graph theoretical sense. So quite surprisingly, the ideas of Ref. [20] carry over to this more general case with little modifications. Note that a mere naive embedding of the potential matrices into banded matrices would be insufficient to find the above conclusion, as for every ordering of physical systems, the range of the banded matrices could not be kept finite. We nevertheless state the full proof here for completeness.

We denote by

$$a := \lambda_{\text{min}}(V), \quad b := \|V\| \tag{75}$$

the minimal and maximal eigenvalue of a real symmetric matrix V of finite range. Let a function $f : \mathbb{C} \to \mathbb{C}$ be such that $f\circ\psi$ is analytic in the interior of an ellipse[2] ε_χ, $\chi > 1$, with focii in -1 and 1 and continuous on ε_χ. Furthermore suppose $(f \circ \psi)(z) \in \mathbb{R}$ for $z \in \mathbb{R}$. Here

$$\psi : \mathbb{C} \to \mathbb{C}, \quad \psi(z) = \frac{(b-a)z + a + b}{2}. \tag{76}$$

We can now state the generalization of Ref. [20].

Theorem 6 (Exponential decay of entries of matrix functions) *Let $V = (V_{i,j})$ be a positive real symmetric matrix of finite range m, $V_{i,j} = 0$ for dist$(i,j) > m/2$, let $f : \mathbb{C} \to \mathbb{C}$ be such that it fulfills the above assumptions. Then there exist constants K and q, $0 \leq K$, $0 \leq q < 1$ such that*

$$|[f(V)]_{i,j}| \leq K q^{\textit{dist}(i,j)}, \tag{77}$$

where

$$K := \max\left\{ \|f(V)\|, \frac{2\chi}{\chi - 1} \max_{z\in\varepsilon_\chi} |(f\circ\psi)(z)| \right\}, \tag{78}$$

$$q := \left(\frac{1}{\chi}\right)^{2/m}. \tag{79}$$

[2] We denote its half axes by α and β, $\alpha > 1$, $\beta > 0$, $\alpha > \beta$. It is then completely specified by the parameter $\chi = \alpha + \beta$. Note that, if α is known so is β as $1 = \alpha^2 - \beta^2$.

Proof. For a function $g : \mathbb{C} \to \mathbb{C}$ analytic in the interior of the ellipse ε_χ, $\chi > 1$, continuous on ε_χ, and with $g(z) \in \mathbb{R}$ for real z, one has

$$\inf\left\{ \max_{-1\leq z\leq 1} |g(z) - p(z)| : p \in P_k \right\} \leq \frac{2}{\chi^k(\chi-1)} \max_{z\in\varepsilon_\chi} |g(z)|, \tag{80}$$

where P_k denotes the set of all polynomials with real coefficients and degree less than or equal to k. This result is due to Bernstein (see Ref. [29] for a proof) and applies to the function $f \circ \psi$. As V is assumed to be of range m, we have that $p(\psi^{-1}(V))$ is of range km for all polynomials $p \in P_k$. Thus, for dist$(i, j) > km/2$ we have

$$(p(\psi^{-1}(V)))_{i,j} = 0. \tag{81}$$

For given $i \neq j$ we now choose k such that

$$k = \lceil 2\text{dist}(i, j)/m \rceil - 1, \tag{82}$$

i.e., we have $k < 2\text{dist}(i, j)/m \leq k + 1$, which yields

$$\begin{aligned} |(f(V))_{i,j}| &= \left|((f\circ\psi)(\psi^{-1}(V)))_{i,j} - (p(\psi^{-1}(V)))_{i,j}\right| \\ &\leq \left\|(f\circ\psi)(\psi^{-1}(V)) - p(\psi^{-1}(V))\right\| \\ &= \max_{-1\leq z\leq 1} |(f\circ\psi)(z) - p(z)|, \end{aligned} \tag{83}$$

where the last equation follows from the fact that the spectrum of $\psi^{-1}(V)$ is contained in the interval $[-1, 1]$. Applying Bernstein's theorem, we know that there exists a sequence of polynomials $p_{(n)}$ of degree k that satisfy

$$\begin{aligned} &\lim_{n\to\infty} \max_{-1\leq z\leq 1} \left|(f\circ\psi)(z) - p_{(n)}(z)\right| \\ &= \inf\left\{ \max_{-1\leq z\leq 1} |(f\circ\psi)(z) - p(z)| : p \in P_k \right\} \\ &\leq \frac{2}{\chi^k(\chi-1)} \max_{z\in\varepsilon_\chi} |(f\circ\psi)(z)| \\ &\leq \frac{2\chi}{\chi-1} \max_{z\in\varepsilon_\chi} |(f\circ\psi)(z)| \left(\frac{1}{\chi}\right)^{2\text{dist}(i,j)/m}. \end{aligned} \tag{84}$$

where the last inequality follows from the choice of k and $\chi > 1$. Along with the fact that $(f(V))_{i,i} \leq \|f(V)\|$ this concludes the proof. □

We will now apply Theorem 3 to the function f defined as

$$f(z) = \frac{2}{\mathrm{e}^{2\sqrt{z}/T} - 1}, \tag{85}$$

– reflecting the matrix function that is needed in order to evaluate the second moments of a Gibbs state – and the matrix $V_x V_p$ in order to prove Theorem 2. In the notation of

Theorem 3 we have $V = V_x V_p$, which is of range $2m$, $a = (\Delta E/2)^2$, and $b = \|V_x V_p\|$. The function $f(z)$ is analytic for all

$$z \in \mathbb{C} \backslash \{z \in \mathbb{R} : z \leq 0\} \tag{86}$$

and therefore $f \circ \psi$ is analytic for all $z \in \mathbb{C}$ with $\Re(z) < (a+b)/(b-a)$. So for $\chi = b/(b-a) = \alpha + \beta$ the function f fulfills the requirements of Theorem 3 and $|(f \circ \psi)|$ attains its maximum at $z = -\alpha$. Thus,

$$\begin{aligned} \frac{2\chi}{\chi - 1} \max_{z \in \varepsilon_\chi} |(f \circ \psi)(z)| &\geq \frac{2}{\mathrm{e}^{2(a(1-a/(4b)))^{1/2}/T} - 1} \\ &\geq \frac{2}{\mathrm{e}^{2\sqrt{a}/T} - 1} = \|f(V)\|. \end{aligned} \tag{87}$$

To summarize, we have

$$|G_{i,j}| \leq \frac{4\|V_x V_p\| / \left(\Delta E/2\right)^2}{\exp\left(\frac{\Delta E}{T}\left(1 - \frac{(\Delta E/2)^2}{4\|V_x V_p\|}\right)^{1/2}\right) - 1} \left(1 - \frac{\left(\Delta E/2\right)^2}{\|V_x V_p\|}\right)^{\mathrm{dist}(i,j)/m}. \tag{88}$$

To bound the entries of $(V_x V_p)^{-1/2}$ we could also apply Theorem 3, however, for this special function, the bound can be given in a more straight-forward manner. To this end, consider again the power series expansion of the square root, Eq. (52). Denote by a and b the minimal and maximal eigenvalue of $V = V_x V_p$, respectively. As matrices V_x and V_p are assumed to be of range m, V is of range $2m$. Thus, for given $i \neq j$ choosing $k = \lceil \mathrm{dist}(i,j)/m \rceil - 1$, i.e., $k < \mathrm{dist}(i,j)/m \leq k+1$, yields

$$\begin{aligned} \left|\left(V^{-1/2}\right)_{i,j}\right| &= \frac{1}{\sqrt{b}} \left| \left((V/b)^{-1/2}\right)_{i,j} - \left(\mathbb{1} + \sum_{r=1}^{k} a_r (\mathbb{1} - V/b)^k\right)_{i,j} \right| \\ &\leq \frac{1}{\sqrt{b}} \left\| (V/b)^{-1/2} - \left(\mathbb{1} + \sum_{r=1}^{k} a_r (\mathbb{1} - V/b)^k\right) \right\| \\ &= \frac{1}{\sqrt{b}} \max_{a/b \leq x \leq 1} \left| \sum_{r=k+1}^{\infty} a_r (1-x)^k \right| \leq \frac{1}{\sqrt{b}} \sum_{r=k+1}^{\infty} a_r (1 - a/b)^k \\ &= \frac{\sqrt{b}}{a} \left(1 - \frac{a}{b}\right)^{k+1} \leq \frac{\sqrt{b}}{a} \left(1 - \frac{a}{b}\right)^{\mathrm{dist}(i,j)/m}. \end{aligned} \tag{89}$$

For diagonal terms, we have $|(V^{-1/2})_{i,i}| \leq \|V^{-1/2}\| = 1/(\Delta E/2)$, i.e., for all i,j we have

$$\left| \left((V_x V_p)^{-1/2}\right)_{i,j} \right| \leq \frac{\sqrt{\|V_x V_p\|}}{(\Delta E/2)^2} \left(1 - \frac{(\Delta E/2)^2}{\|V_x V_p\|}\right)^{\mathrm{dist}(i,j)/m}, \tag{90}$$

and thus, together with the assumption on the lattice,

$$\left| ((V_x V_p)^{-1/2} G)_{i,j} \right| \leq \frac{4 l_0 \|V_x V_p\|^{3/2} / \left(\Delta E/2\right)^4}{\exp\left(\Delta E \left(1 - \frac{(\Delta E/2)^2}{4\|V_x V_p\|}\right)^{1/2} / T\right) - 1} \mathrm{e}^{-\mathrm{dist}(i,j)\nu}. \tag{91}$$

The last step is now analogous to the proof in the zero temperature case, Eq. (64). We finally arrive at

$$\left|\left(V_p(V_xV_p)^{-1/2}G\right)_{i,j}\right| \leq \frac{4l_0\|V_xV_p\|^{3/2}v_{d,m/2}\|V_p\|/\left(\Delta E/2\right)^4}{\exp\left(\Delta E\left(1-\frac{(\Delta E/2)^2}{4\|V_xV_p\|}\right)^{1/2}/T\right)-1}\mathrm{e}^{-\nu\mathrm{dist}(i,j)/2}, \quad (92)$$

$$\left|\left(V_x(V_xV_p)^{-1/2}G\right)_{i,j}\right| \leq \frac{4l_0\|V_xV_p\|^{3/2}v_{d,m/2}\|V_x\|/\left(\Delta E/2\right)^4}{\exp\left(\Delta E\left(1-\frac{(\Delta E/2)^2}{4\|V_xV_p\|}\right)^{1/2}/T\right)-1}\mathrm{e}^{-\nu\mathrm{dist}(i,j)/2} \quad (93)$$

for $\mathrm{dist}(i,j) \geq m$, which concludes the proof. □

3.5 Proof of Theorem 5

From entanglement theory we know that an upper bound to the entropy of entanglement is given by the logarithmic negativity [38, 39, 40, 41, 42, 43], which in turn can be bounded from above by an expression depending only on the two-point position correlation function with respect to the ground state [12],

$$E_S^I \leq \frac{4\|V\|^{1/2}}{\log(2)} \sum_{\substack{i\in I\\ j\in L\backslash I}} |\langle \hat{x}_i\hat{x}_j\rangle| \,. \quad (94)$$

From the proof of Theorem 4, Eq. (90), we know that

$$|\langle \hat{x}_i\hat{x}_j\rangle| = \left|\left(V^{-1/2}\right)\right| \leq \frac{\|V\|^{1/2}}{(\Delta E/2)^2}\exp[-\mathrm{dist}(i,j)/\xi], \quad (95)$$

where

$$\xi = \frac{m}{\log\left(\frac{\|V\|}{\|V\|-(\Delta E/2)^2}\right)}. \quad (96)$$

Thus, we find

$$\log(2)E_S^I \leq \frac{4\|V\|}{(\Delta E/2)^2}\sum_{\substack{i\in I\\ j\in L\backslash I}} \exp[-\mathrm{dist}(i,j)/\xi] = \frac{4\|V\|}{(\Delta E/2)^2}\sum_{r=1}^{\infty}\mathrm{e}^{-r/\xi}N_r, \quad (97)$$

where we defined

$$N_r = \sum_{j\in L\backslash I}\ \sum_{\substack{i\in I\\ \mathrm{dist}(i,j)=r}} 1. \quad (98)$$

Note that, N_1 coincides with the definition of the surface area of I, $N_1 = s(I)$. Let us now define the "outer boundary" of I:

$$\partial I := \{j\in L\backslash I : \text{ there exists a } i\in I \text{ such that } \mathrm{dist}(i,j)=1\}\,. \quad (99)$$

We immediately find $|\partial I| \leq s(I)$ and we can restrict the sum over $L\backslash I$ to the set

$$A_r := \bigcup_{i \in \partial I} \{j \in L\backslash I : \text{dist}(i,j) \leq r-1\}\,, \quad |A_r| \leq s(I) v_{d,r}, \tag{100}$$

i.e.,

$$\begin{aligned} N_r &= \sum_{j \in A_r} \sum_{\substack{i \in I \\ \text{dist}(i,j)=r}} 1 \leq \sum_{j \in A_r} \sum_{\substack{i \in L \\ \text{dist}(i,j)=r}} 1 = \sum_{j \in A_r} |S_r(j)| \\ &\leq c r^{d-1} v_{d,r} s(I) \leq c^2 r^{2d-1} s(I). \end{aligned} \tag{101}$$

To summarize,

$$\begin{aligned} E_S^I &\leq \frac{4\|V\|c^2}{\log(2)(\Delta E/2)^2} s(I) \sum_{r=1}^{\infty} \mathrm{e}^{-r/\xi} r^{2d-1} \\ &= \frac{4\|V\|c^2 Li_{1-2d}(\mathrm{e}^{-1/\xi})}{\log(2)(\Delta E/2)^2} s(I), \end{aligned} \tag{102}$$

where Li_{1-2d} is the polylogarithm of degree $1-2d$. □

4 Discussion and examples

In this section, we discuss a few special cases to exemplify the previous results. It is important to note that the previous results hold true in case of general lattices, beyond structures with a very high translational symmetry.

Example 1 (Disordered one-dimensional system) *This example is to highlight that besides locality, no assumptions on the coupling are required. In particular, we allow for random coupling, as for example in the following sense. Consider a sequence of one-dimensional systems on a one-dimensional chain with periodic boundary conditions, such that* $G^{(n)} = (L^{(n)}, E^{(n)})$ *is characterized by* $L^{(n)} = [1, ..., n]$ *and the adjacency matrix with entries*

$$E_{i,j}^{(n)} = \delta_{i,j+1} + \delta_{i,j-1} + \delta_{i,n}\delta_{j,1} + \delta_{j,n}\delta_{i,1}, \quad i,j = 1, ..., n. \tag{103}$$

The coupling is specified by $C^{(n)} = (G^{(n)}, V^{(n)}, \mathbb{1})$*. Now let* $(r_1, ..., r_n)$ *be a vector of realizations of random numbers taken from the interval* $[0,1]$*. Now let* $(V^{(n)})_{i,i} = 3$*, and*

$$(V^{(n)})_{i,j} = (\delta_{i,j+1} + \delta_{i,j-1})\, r_i + (\delta_{i,n}\delta_{j,1} + \delta_{j,n}\delta_{i,1})\, r_n, \qquad i,j = 1, ..., n. \tag{104}$$

From Gershgorin's theorem we then know that $\Delta E = 2\lambda_{min}^{1/2}(V_x^{(n)}) \geq 2$*. Also,* $\|V^{(n)}\| \leq 5$ *for all* n*. Hence, Theorem 1 can be applied, and we find exponentially decaying correlation functions, even in this disordered system. Note that this example can also be generalized to* d*-dimensional general lattices, as one can straightforwardly find upper and lower bounds to the spectral values of* V *using the same arguments as above.*

Example 2 (Thermal states of rotating wave Hamiltonians) *Consider the coupling $C^{(n)} = (G^{(n)}, V^{(n)}, V^{(n)})$ on a general graph $G^{(n)} = (L^{(n)}, E^{(n)})$. Such Hamiltonians correspond to the case of "rotating-wave Hamiltonians". For each n, the ground state is easy to find: it is the product state of uncoupled degrees of freedom. The covariance matrix for the zero temperature case is then given by*

$$\gamma = \mathbb{1} \oplus \mathbb{1} \tag{105}$$

for all n. Indeed, the correlation length is zero. For finite temperature however, we find

$$\gamma(T) = \left(\mathbb{1} + 2\left(\exp(2V^{(n)}/T) - \mathbb{1}\right)^{-1}\right) \oplus \left(\mathbb{1} + 2\left(\exp(2V^{(n)}/T) - \mathbb{1}\right)^{-1}\right). \tag{106}$$

Now, e.g., choose

$$V^{(n)} = \mathbb{1} - cE^{(n)}, \tag{107}$$

where $0 < c < 1/2$ and $E^{(n)}$ as in the previous example. Due to the circulant structure of $V^{(n)}$, we have $\Delta E = 2\sqrt{1-2c}$, $\|V\| = 1 + 2c$, and

$$\langle \hat{x}_i \hat{x}_j \rangle = \langle \hat{p}_i \hat{p}_j \rangle = \delta_{i,j} + \frac{2}{n} \sum_{k=1}^{n} \frac{\mathrm{e}^{2\pi k(i-j)/n}}{\mathrm{e}^{2\lambda_k(V)/T} - 1}. \tag{108}$$

Subsequently, we merely sketch the argument leading to the statement that the correlation length is non-zero for $T > 0$, but it should be clear how this can be made a rigorous statement. For high temperature we can approximate $\exp(2\lambda_k(V)/T) - 1 \approx 2\lambda_k(V)/T$, and thus

$$\langle \hat{x}_i \hat{x}_j \rangle = \langle \hat{p}_i \hat{p}_j \rangle \approx \delta_{i,j} + T((V^{(n)})^{-1})_{i,j}. \tag{109}$$

The explicit inverse of $V^{(n)}$ is known [30] and given by

$$((V^{(n)})^{-1})_{i,j} = \frac{1+q^2}{(q^n-1)(q^2-1)} \left(q^{n-|i-j|} + q^{|i-j|}\right), \tag{110}$$

$$q := \frac{1-\sqrt{1-4c^2}}{2c}. \tag{111}$$

So, in the limit $n \to \infty$, n odd, we get for $i \neq j$

$$\langle \hat{x}_i \hat{x}_j \rangle = \langle \hat{p}_i \hat{p}_j \rangle = \frac{1+q^2}{1-q^2} T \exp(-\mathit{dist}(i,j)/\xi) \quad \textit{for} \quad T \gg 1, \tag{112}$$

where

$$\xi = \frac{1}{\log(1/q)} > 0. \tag{113}$$

Hence, the thermal states have longer-ranged correlations as compared to the ground state. This can be explained as the first excited state is already no longer a product state, unlike the ground state itself.

Example 3 (Energy gap from exponential decay) *Here we will exemplify the argument leading to a statement on the existence of a spectral gap for a simple one-dimensional system. To this end consider again the one dimensional chain* $G^{(n)} = (L^{(n)}, E^{(n)})$ *with* $L^{(n)}$ *and* $E^{(n)}$ *as above and* $C^{(n)} = (G^{(n)}, V^{(n)}, \mathbb{1})$*, i.e.,* $\Delta E^{(n)} = 2\lambda_{min}^{1/2}(V^{(n)})$*. Now suppose that in the large system limit,* $n \to \infty$*,* n *odd, either*

$$\langle \hat{x}_i \hat{x}_j \rangle = K \exp(-dist(i,j)/\xi) \tag{114}$$

or

$$\langle \hat{p}_i \hat{p}_j \rangle = K \exp(-dist(i,j)/\xi), \tag{115}$$

with constants $K > 0$ *and* $\xi > 0$*. We can then show that the system is gapped in the following way: Consider the matrix* A^{-1} *(cp. the previous example) the entries of which are given by*

$$(A^{-1})_{i,j} = \frac{1+q^2}{(q^n-1)(q^2-1)} \left(q^{n-|i-j|} + q^{|i-j|} \right). \tag{116}$$

That is, if (114) holds, we have in the limit of large n

$$\langle \hat{x}_i \hat{x}_j \rangle = K \frac{1-q^2}{1+q^2} (A^{-1})_{i,j}, \tag{117}$$

with $0 < q = \exp(-1/\xi) < 1$*. Under (115) the same holds for* $\langle \hat{p}_i \hat{p}_j \rangle$*. As we know that the inverse of* A *is a circulant matrix,* $A = \mathbb{1} - cE^{(n)}$*,* $c = q/(1+q^2)$*, we know the minimal and maximal eigenvalue of* A *are given by* $\|A^{-1}\| = 1+2c$*,* $\lambda_{min}(A) = 1-2c$ *for all* n*. And thus, under (114)*

$$\begin{aligned} \Delta E &= 2\lambda_{min}(V^{1/2}) = \frac{2}{\lambda_{max}(V^{-1/2})} \\ &= \frac{2(1+q^2)}{K(1-q^2)\lambda_{max}(A^{-1})} \\ &= \frac{2(1-q)^2}{K(1-q^2)}. \end{aligned} \tag{118}$$

Similarly, in case that (115) holds true, we arrive at

$$\begin{aligned} \Delta E &= 2\lambda_{min}(V^{1/2}) = 2K\frac{1-q^2}{1+q^2}\lambda_{min}(A^{-1}) \\ &= \frac{2K(1-q^2)}{(1+q)^2}. \end{aligned} \tag{119}$$

We can hence conclude in both cases that we have that the system is gapped,

$$\Delta E \geq \frac{2(1-\exp(-1/\xi))^2}{1-\exp(-2/\xi)} \min\{K, 1/K\} > 0. \tag{120}$$

Example 4 (Algebraically decaying correlation functions) *Consider a general lattice $G^{(n)}$ of dimension d equipped with the coupling $C^{(n)} = (G^{(n)}, V^{(n)}, \mathbb{1})$, where we define the translationally invariant coupling $V^{(n)}$ via, $\eta > d$,*

$$(V^{(n)})^{-1/2} = W^{(n)}, \quad (W^{(n)})_{i,j} = \begin{cases} \mathit{dist}(i,j)^{-\eta} & \text{for } i \neq j, \\ 1 + \sum_{\substack{j \in L \\ j \neq i}} \mathit{dist}(i,j)^{-\eta} & \text{for } i = j. \end{cases} \tag{121}$$

Again, from Gershgorin's theorem, we know that $\lambda_{min}(W^{(n)}) \geq 1$ and, together with the definition of the lattice dimension, Eq. (3),

$$\begin{aligned} \|W^{(n)}\| &\leq 1 + 2 \sum_{\substack{j \in L \\ j \neq i}} \frac{1}{\mathit{dist}(i,j)^{\eta}} \\ &\leq 1 + 2 \sum_{r=1}^{\infty} r^{-\eta} |S_r(i)| \\ &\leq 1 + 2c \sum_{r=1}^{\infty} r^{d-1-\eta} \\ &= 1 + 2c\zeta(1 - d + \eta) < \infty, \end{aligned} \tag{122}$$

for all n. The system is thus gapped and $\|V^{(n)}\| \leq 1$, i.e., all assumptions of Theorem 1 are met except that $V^{(n)}$ is not of finite range. Hence we cannot conclude that the correlation functions are exponentially decaying. Indeed, we find that the position correlation function for $i \neq j$ is given by

$$\langle \hat{x}_i \hat{x}_j \rangle = \frac{1}{\mathit{dist}(i,j)^{\eta}}, \tag{123}$$

so the correlations in this non-local system are algebraically decaying, despite being gapped.

5 Outlook

In this paper, we have rigorously revisited the question of algebraically and exponentially decaying correlation functions in gapped harmonic systems on general lattices. Both for the ground states, as well as for Gibbs states, we considered the implications of the gap and the coupling strength to the correlation length. For systems only coupled in position, we showed that an energy gap can be deduced from algebraically decaying correlations. We also found an equivalence between the existence of a gap and exponentially decaying correlation functions for local Hamiltonians. For zero temperature, no assumptions have been made on the underlying lattice. This harmonic quasi-free case is a particularly transparent instance of a physical system where this physically plausible connection can be explored, explicitly making use of properties of matrix functions of banded matrices, instead of exploiting Lieb-Robinson type results. As such, we cover cases not included in the class of systems considered in Ref. [4].

We showed that these statements have immediate implications on the scaling of entanglement in ground and thermal states of quantum many-body systems: under these conditions, the geometric entropy, so the von-Neumann entropy of the reduction, is bounded from above by a quantity that is linear in the boundary area of the distinguished region. Equivalently, this quantifies the degree of entanglement of the region with respect to the rest of the lattice. This result on a connection between the surface area and the degree of entanglement can hence also be identified on general graphs, hence establishing an rigorous entanglement-area relationship in general finite-ranged gapped harmonic lattice systems.

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